A frequency approach to topological identification and graphical modeling

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Aim of the work

The work illustrates recent developments in dynamical network analysis.

Systems of interconnected agents have been largely studied in a wide range of fields, because

- they are naturally suited to describe *spatially distributed processes*
- a large number of real systems can be described via *networked discrete elements*
- *neural networks* have been proved to be able to model a wide range of complex functions
- they are a powerful tool to model large scale systems by dividing them into sub-processes of reduced complexity.

Remark

The *topology* results of fundamental importance in the definition of the properties of the final network.
Not very interesting networks ...

Usually the link structure is already given or it is the objective of the design; thus, methods to infer the connections of a given network have only recently started to be investigated.

The actual results in the literature always assume at least one of the following hypotheses:

- the model/dynamics of each single agent in the network is already known (with a certain degree of certainty);
- the network behaviour can be tested via forcing external input signals.

The above assumptions are quite reasonable, when the scenario concerns human-made networks and, then, the topological identification problem is marginal or even not interesting at all. However, they turn infeasible for a large of real networks, especially regarding biological and social processes.
Problem

Given a network of interconnected agents, find the internal link structure under the following assumptions:

- the network can not be manipulated, for instance, via probing or testing input signals;
- the agents belong to a certain class of systems, but their exact model or even an approximation of that is unknown;
- the connections are defined in a specific class of mathematical functions, though unknown;
- the (scalar) outputs of each agent are the only observable measures;
- the topology belongs to a certain class of graphs.

This problem is sure a formidable one, but in the following it will be illustrated how it can be solved under proper choices for the above features.
In the rest of the work the following scenario is considered.

**Acyclic Linear Networks:**
- each agent is described by an unknown linear Ordinary Differential Equation model
- every link acts on a scalar output as an unknown linear operator, i.e., via a certain Transfer Function
- inputs are computed via additive models
- the directed graph describing the network topology is a polytree, i.e., an acyclic graph with neither undirected nor directed closed paths.
The problem set up

In the linear framework each agent admits the following description

\[ X_j(z) = e_j(z) + \sum_{i \in \mathcal{I}(j)} W_{ji}(z) X_i(z) \]

where

- \( X_j(z) \) is the Z-transform obtained from the samples of the \( j \)-th agent’s output, once this series has been deprived of any deterministic component in order to be considered a wide-sense stationary random process with zero mean (the same regarding \( X_i(z) \))
- \( W_{ji}(z) \) is the transfer function, which describes the functional dependency of the \( j \)-th agent from the \( i \)-th one
- \( \mathcal{I}(j) \) denotes the set of all the inputs processes \( \{ X_\tau \}_{\tau \in \mathcal{I}(j)} \), which affects the \( j \)-th one
- \( e_j(z) \) represents the noise affecting the measurements.
The MISO Wiener filtering approach ...

Observation

The previous formulation admits the application of the Multiple Input Single Output Wiener filter in order to derive the set of transfer functions $\hat{W}_{ji}(z)$ minimizing the mean quadratic error $E[\varepsilon_j^2]$, being

$$\varepsilon_j(z) := X_j(z) - \sum_{i \in \hat{I}(j)} \hat{W}_{ji}(z)X_i(z)$$

Since $I(j)$ is the very objective of the topological identification the following question arises:

why do not obtain $I(j)$ as a by-product of a MISO Wiener filter, in term of the optimal set $\hat{I}(j)$?
... fails!

Proposition

The MISO Wiener filter between $X_j$ and all the other processes $\{X_i\}_{i \neq j}$ may have non zero components, i.e. elements $W_{ji}(z)$ different from the zero transfer function, only with respect to the signals belonging to its Markov Blanket\(^1\).

Conclusion

Multiple applications of the MISO Wiener filter are not able, just by themselves, to recover the correct topology of the network. In particular, this approach doesn’t ensure that the identified set $\{\hat{I}(j)\}_j$ describes an acyclic network.

As a further drawback, the computational burden to compute the MISO Wiener filter for each agent of the network may turn out infeasible in a large scale system.

\(^1\): in a directed graph the Markov Blanket of a node is the ensemble of its parents, its children and all the parents of its children.
The coherence-based approach

In this work an hybrid approach, combining frequency domain identification techniques and graph properties, is illustrated.

Proposition

The weighted mean quadratic error $E[\varepsilon_j^2]$ of the Single Input Single Output Wiener filter, where

$$\varepsilon_j(z) := Q_j(z) \left( X_j(z) - \hat{W}_{ji}(z) X_i(z) \right),$$

for a proper choice of the dynamical weighting function $Q_j(Z)$ boils down to

$$d(X_i, X_j) := \sqrt{\min_{W_{ji}(Z)} E[\varepsilon_j^2|Q_j]]} = \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - C_{X_i X_j}(\omega)) d\omega \right]^{1/2}$$

and depends on the coherence function $^2 C_{X_i X_j}(\omega)$ between $X_j$ and $X_i$.

$^2$: the coherence function extends the concept of correlation and standard efficient algorithms for its computation exist.
The Minimum Spanning Tree extraction

**Proposition**

The previous operator $d(\cdot, \cdot)$ defines a **pseudo-metric** for the set $\{X_i\}$.

The above result allows one to define the **distance matrix** $D(j, i) := d(X_j, X_i)$ for the considered network.

**Main theorem**

The **Minimum Spanning Tree** extracted from the distance matrix $D$ is equal to the undirected version of the **actual connection graph** of the original acyclic linear network.

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3: a **MST** consists of a tree graph connecting all the nodes and minimizing the total weight of the connections.
Standard identification techniques fail in detecting the real topology even in the linear framework.

A mixed approach combining identification and graph theory correctly solves the problem.

The computation of the coherence functions and the extraction of the MST is, in principle, lighter than solving multiple MISO Wiener filter problems.

The topological identification problem has strong connections to the graphical models theory, i.e., to Bayesian Networks and Markov Fields.

From the modelling point of view a causal version of the illustrated approach can be derived with some limitations.

The same problem can be interpreted in a Compressive Sensing scenario as an approach to point out a reasonably small basis aimed to represent a whole set of processes.
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Any deterministic component has been removed from the hourly times series of the temperatures.

The distance matrix $D$ has been computed.

The MST has been extracted from $D$.

The resulting graph has been found reasonable and coherent with expectations.
Example: Graphical models of European climatic regions

Figure: (solid + dotted lines) the MST over the measurement sites; (solid line) minimum cost edge for at least one of the connected nodes.